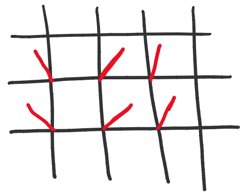
More on graph expansion

Monday, May 31, 2021   11:10 PM

Duality between two different models : XY model and SOS model. [Ref: Ch 4.6.2 of Mussardo]

XY model on square lattice:

$$H = -J\sum_{\langle r,r'\rangle}\cos(\theta_r-\theta_{r'})$$



Partion function

$$Z = \int_{-\pi}^{\pi}\prod_m \frac{d\theta_m}{2\pi}\prod_{\langle r,r'\rangle} e^{J\cos(\theta_r-\theta_{r'})}$$

Using Fourier transformation

$$e^{J\cos\theta} = \sum_{n=-\infty}^{\infty} e^{f(n)}\, e^{2\pi i n\theta}$$

with

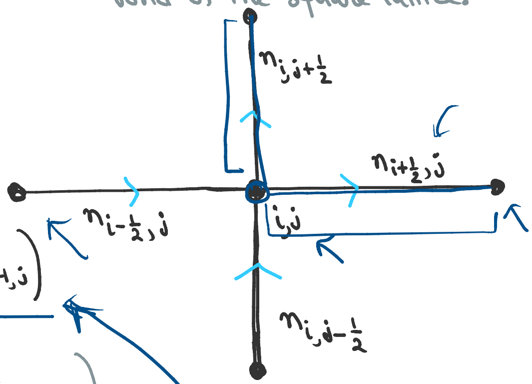
$$e^{f(n)} = \int_{-\pi}^{\pi}\frac{d\theta}{2\pi}\, e^{J\cos\theta}\, e^{-2\pi i n\theta}$$

note $f(-n)=f(n)$

gives

$$Z = \int_{-\pi}^{\pi}\prod_m\frac{d\theta_m}{2\pi}\prod_{\langle r,r'\rangle}\sum_{n_{r,r'}=-\infty}^{\infty} e^{f(n_{r,r'})}\, e^{2\pi i n_{r,r'}(\theta_r-\theta_{r'})}$$

$n_{r,r'}$ is defined along each bond of the square lattice.

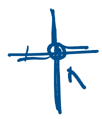


① $\prod_{\langle r,r'\rangle} e^{f(n_{r,r'})}\, e^{2\pi i n_{r,r'}(\theta_r-\theta_{r'})}$

$$= \prod_{i,j} e^{f(n_{i+\frac{1}{2},j}) + 2\pi i n_{i+\frac{1}{2},j}(\theta_{i,j}-\theta_{i+1,j})}\; e^{f(n_{i,j+\frac{1}{2}}) + 2\pi i n_{i,j+\frac{1}{2}}(\theta_{i,j}-\theta_{i,j+1})}$$

$$= \prod_{ij} e^{f(n_{i-\frac{1}{2},j}) + f(n_{i+\frac{1}{2},j}) + 2\pi i\,\theta_{ij}(-n_{i-\frac{1}{2},j}+n_{i+\frac{1}{2},j})}\; e^{f(n_{i,j-\frac{1}{2}})} + \cdots$$

$$= \prod_{ij} e^{f(n_{i-\frac{1}{2},j}) + f(n_{i+\frac{1}{2},j}) + f(n_{i,j-\frac{1}{2}}) + f(n_{i,j+\frac{1}{2}})}\; e^{2\pi i\,\theta_{ij}\{n_{i+\frac{1}{2},j} - n_{i-\frac{1}{2},j} + n_{i,j+\frac{1}{2}} - n_{i,j-\frac{1}{2}}\}}$$



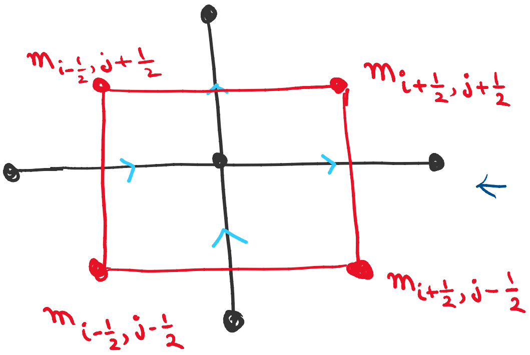
Doing the θ integral with $\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\, e^{2\pi i\theta n} = \delta_{n,0}$

gives

$$Z = \sum_{\{\bar{n}\}} e^{f(\bar{n})}$$ of all integer configurations on the bonds such that at each site $\bar{n}$ is divergence free

$$\boxed{n_{i+\frac{1}{2},j} - n_{i-\frac{1}{2},j} + n_{i,j+\frac{1}{2}} - n_{i,j-\frac{1}{2}} = 0}$$

Such a "divergence" free bond vectors we can express as "curl" of integers on dual lattice



$$n_{i+\frac{1}{2},j} = m_{i+\frac{1}{2},j+\frac{1}{2}} - m_{i+\frac{1}{2},j-\frac{1}{2}}$$

$$n_{i,j+\frac{1}{2}} = -m_{i+\frac{1}{2},j+\frac{1}{2}} + m_{i-\frac{1}{2},j+\frac{1}{2}}$$

Then, further using $f(-n) = f(n)$ we get

$$Z = \underbrace{\sum_{\{m\}}}_{\text{all integers}} \sum_{\langle r,r'\rangle} e^{f(m_r - m_{r'})}$$ on dual lattice

This is the SOS model on square lattice. Important for surface growth.

Series expansion of Potts model and relation to percolation and four colour problem

$$Z_N = \sum_{\{\sigma\}} \prod_{\langle i,j\rangle} e^{J\delta_{\sigma_i\sigma_j}} = \sum_{\{\sigma\}} \prod_{\langle i,j\rangle} \left[1 + \delta_{\sigma_i\sigma_j}\underbrace{(e^J - 1)}_{v}\right]$$ $\sigma = 1, 2, \cdots, q$

$$\sum_{\sigma_1}\sum_{\sigma_2} \delta_{\sigma_1\sigma_2}$$

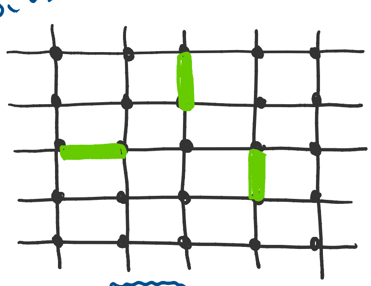
$$= q^N + q^{N_0} q^{N_1} v + \cdots$$

$$= \sum_G q^c v^l$$

all graphs on the lattice

c: Total number of connected clusters on G. An isolated site is considered as a single cluster $(c=1)$

l: number of links on G.



$c = \widetilde{N-6} + 3$

$l = 1 + 1 + 1$

Note: unlike in Ising model, open end graphs also contribute.

Remark: there is a similar low temperature expansion and a high T - low T duality.

Important limits: (1) $J = -\infty$ limit. Only single cluster graphs contribute, (all neighbors have distinct states)

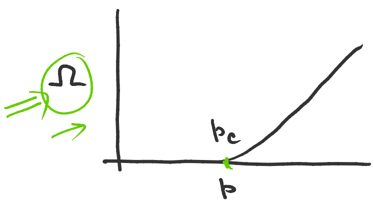
In this case Z_N gives the number of distinct ways the lattice nodes can be colored with q colors so that no two neighbors have same color. This is called chromatic polynomial $P_N(q)$ of a lattice or graph.

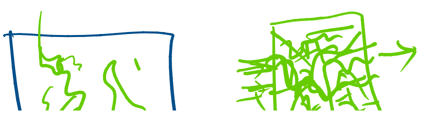
$$P_N(q) = \lim_{v \to 0} Z_N(v)$$

Remark: for planar two-dimensional graphs the above limit has a relation to four-color problem. It roughly states that any geographical planar map can be coloured using four colours such that no two neighboring nations have same color.
This amounts to showing that $P_N(q)$ has no roots at $q = 4$.
Exact mathematical problem is more detailed
See Appendix 2C of Mussardo.

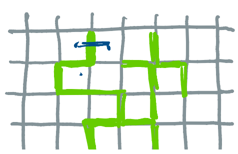
(2) limit $q \to 1$. Statement: for $q \to 1$, Z_N gives different configurations of bond percolation with $p = \frac{v}{1-v}$.

Bond percolation: An edge is connected with prob p. There will be many connected clusters. Question asked, what is the probability to have an infinite cluster? This prob jumps from 0 to 1 at a critical value p_c. The average fraction of sites belonging to this infinite cluster Ω grows continuously as $(p-p_c)^\beta$ for $p \geq p_c$.
Mean size of connected clusters grow as $(p_c - p)^{-\gamma}$ for $p < p_c$.
Percolation is one of the simplest examples of continuous phase transition, and these exponents (α, β) are critical exponents.

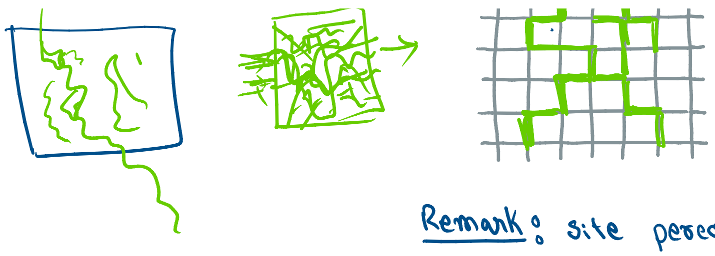




Remark: on square lattice $p_c = \frac{1}{2}$ for a bond percolation. This can be seen from duality.



For a cluster config on square lattice there is a cluster config on dual lattice where prob $p \to 1-p$.



there is a cluster config on dual lattice where prob $p \to 1-p$.
Then self duality of square lattice gives
$$p_c = \frac{1}{2}.$$

Remark: site percolation is where a site is filled or not with probability p and $1-p$.

Relation to Potts model:

$$Z_{Potts} = \sum_G v^\ell q^c = (1-p)^{-M} \sum_G p^\ell (1-p)^{M-\ell} q^c, \qquad v = \frac{p}{1-p}$$

$M \equiv$ total no. of edges on the lattice.

$$= (1-p)^{-M} \langle q^c \rangle$$

average over bond percolation config.

[In bond percolation prob of a graph G is $\pi(G) = p^\ell (1-p)^{M-\ell}$]

Then, for $q \to 1$

$$\boxed{Z_{Potts} = (1-p)^{-M} Z_{percolation}}$$

Ref: Percolation and the Potts model. F.Y. Wu, J Stat Phys, 18, 1978.

(not exact solution)
A rough summary chart of phase transitions on lattice models with short-range ferromagnetic interactions.

		$d=1$	$d=2$	$d \geq 3$
Ising		No PT	cont PT	cont PT
Potts	$q \leq 4$	No PT	cont PT	cont PT
	$q \geq 5$	No PT	discont PT	discont PT
$O(n)$	$n=2$	No PT	BKT PT	cont PT
	$n \geq 3$	No PT	No PT	cont PT
Z_n	$2 \leq n \leq 4$	No PT	BKT	
	$n \geq 5$	no PT	a non-trivial BKT	